

# New trigonometric identities and reciprocity laws of generalized Dedekind sums

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## Abstract

We obtain new trigonometric identities, which are some product-to-sum type formulas for the higher derivatives of the cotangent and cosecant functions. Further, from specializations of our formulas, we derive not only various known reciprocity laws of generalized Dedekind sums but also new reciprocity laws of generalized Dedekind sums.

## 1 Introduction

From Dedekind, so-called *Dedekind sums* and their reciprocity laws have been studied by the distinguished mathematicians. For an overview of previously defined generalized Dedekind sums, we refer to a good interpretation by M. Beck (see Section 1 and 2 in [1]). Let  $\cot^{(m)}$  denote the  $m$ -th derivative of the cotangent function. In [1], for  $a_0, a_1, \dots, a_r, m_0, m_1, \dots, m_r \in \mathbb{Z}_{\geq 1}, w_0, w_1, \dots, w_r \in \mathbb{C}$ , Beck introduced Dedekind cotangent sums

$$\frac{1}{a_0^{m_0}} \sum_{k \bmod a_0} \prod_{j=1}^r \cot^{(m_j-1)} \left( \pi \left( a_j \frac{k + w_0}{a_0} - w_j \right) \right),$$

where the sum is taken over  $k \bmod a_0$  for which the summand is not singular. The Dedekind cotangent sums include as special cases various generalizations of Dedekind sums expressed by the cotangent functions and their higher derivatives. Moreover, under some conditions for  $a_0, \dots, a_r, w_0, \dots, w_r$ , Beck computed the residue of

$$\cot^{(m_0-1)}(\pi(a_0 z - w_0)) \prod_{l=1}^r \cot^{(m_l-1)}(\pi(a_l z - w_l))$$

and derived various reciprocity laws of the Dedekind cotangent sums, which are not only known results by Dedekind, Rademacher, Apostol, Carlitz, Mikolás, Dieter, Zagier, but also the truly new ones. However, since his method needs a case analysis based on some conditions for singular points of an integrand function of residue calculus, we have to prove the reciprocity laws individually.

On the other hand, an analogue of the Dedekind sum which was formed by replacing the cotangent functions in the Dedekind sum by the cosecant functions, and its reciprocity laws were introduced and proved by Fukuhara [5]. For example, Fukuhara treated the following type formulas. Let  $p$  and  $q$  are relatively prime positive integers.

(0) (Proposition 1.3 in [4] or (1.1) in [5]) For any complex number  $z$ ,

$$\begin{aligned} pq \cot(pz) \cot(qz) &= -\cot^{(1)}(z) - pq + q \sum_{\mu=1}^{p-1} \cot\left(\frac{\pi q \mu}{p}\right) \cot\left(z - \frac{\pi \mu}{p}\right) \\ &\quad + p \sum_{\mu=1}^{q-1} \cot\left(\frac{\pi p \mu}{q}\right) \cot\left(z - \frac{\pi \mu}{q}\right). \end{aligned} \quad (1.1)$$

(1) ((1.2) in [5]) If  $q$  is even, then

$$\begin{aligned} pq \cot(pz) \csc(qz) &= -\cot^{(1)}(z) + q \sum_{\mu=1}^{p-1} \csc\left(\frac{\pi q \mu}{p}\right) \cot\left(z - \frac{\pi \mu}{p}\right) \\ &\quad + p \sum_{\mu=1}^{q-1} (-1)^\mu \cot\left(\frac{\pi p \mu}{q}\right) \cot\left(z - \frac{\pi \mu}{q}\right). \end{aligned} \quad (1.2)$$

(2) ((1.4) in [5]) If  $q$  is odd, then

$$\begin{aligned} pq \cot(pz) \csc(qz) &= -\csc^{(1)}(z) + q \sum_{\mu=1}^{p-1} \csc\left(\frac{\pi q \mu}{p}\right) \csc\left(z - \frac{\pi \mu}{p}\right) \\ &\quad + p \sum_{\mu=1}^{q-1} (-1)^\mu \cot\left(\frac{\pi p \mu}{q}\right) \csc\left(z - \frac{\pi \mu}{q}\right). \end{aligned} \quad (1.3)$$

(3) ((1.3) in [5]) If  $p + q$  is even, then

$$\begin{aligned} pq \csc(pz) \csc(qz) &= -\cot^{(1)}(z) + q \sum_{\mu=1}^{p-1} (-1)^\mu \csc\left(\frac{\pi q \mu}{p}\right) \cot\left(z - \frac{\pi \mu}{p}\right) \\ &\quad + p \sum_{\mu=1}^{q-1} (-1)^\mu \csc\left(\frac{\pi p \mu}{q}\right) \cot\left(z - \frac{\pi \mu}{q}\right). \end{aligned} \quad (1.4)$$

(4) ((1.5) in [5]) If  $p + q$  is odd, then

$$\begin{aligned} pq \csc(pz) \csc(qz) &= -\csc^{(1)}(z) + q \sum_{\mu=1}^{p-1} (-1)^\mu \csc\left(\frac{\pi q \mu}{p}\right) \csc\left(z - \frac{\pi \mu}{p}\right) \\ &\quad + p \sum_{\mu=1}^{q-1} (-1)^\mu \csc\left(\frac{\pi p \mu}{q}\right) \csc\left(z - \frac{\pi \mu}{q}\right), \end{aligned} \quad (1.5)$$

where  $\csc^{(1)}(z)$  is the derivative of  $\csc(z)$ .

He also pointed out that these formulas can be regarded as a one parameter deformation of the reciprocity laws of some Dedekind sums, or a generating function of the reciprocity laws of some Dedekind-Apostol sums. Actually, in (1.1), by comparing the coefficients of the Laurent expansion of (1.1) at  $z = 0$ , we obtain the reciprocity laws of the Dedekind-Apostol sums

$$s_N(q; p) := \frac{1}{2^{N+1}p} \sum_{\mu=1}^{p-1} \cot\left(\frac{\pi q\mu}{p}\right) \cot^{(N-1)}\left(\frac{\pi\mu}{p}\right)$$

as follows.

$$s_1(q; p) + s_1(p; q) = \frac{p^2 + q^2 + 1 - 3pq}{12pq}, \quad (1.6)$$

$$\begin{aligned} s_{2k+1}(q; p) + s_{2k+1}(p; q) &= \frac{1}{2pq} \frac{B_{2k+2}}{k+1} + \frac{B_{2k+2}}{(2k+1)(2k+2)} (p^{2k+1}q^{-1} + p^{-1}q^{2k+1}) \\ &\quad - (2k)! \sum_{l=1}^k \frac{B_{2l}B_{2k+2-2l}}{(2l)!(2k+2-2l)!} p^{2l-1} q^{2k+1-2l}. \end{aligned} \quad (1.7)$$

Here,  $k$  is a positive integer and  $\{B_m\}_{m=0,1,\dots}$  are the Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} t^m.$$

As described above, from some product-to-sum type formulas for some trigonometric functions, like (1.1), we easily obtain the reciprocity laws for various generalized Dedekind sums. In this article, taking into account the investigations, we present a detailed calculation of

$$\prod_{l=1}^{j_I} a_l^{m_l} \cot^{(m_l-1)}(\pi(a_l z - w_l)) \prod_{l=j_I+1}^{j_I+j_{II}} a_l^{m_l} \csc^{(m_l-1)}(\pi(a_l z - w_l))$$

and give a sum expression of the higher derivatives for the cotangent and cosecant functions, which can be regarded as a product-to-sum type formula for the higher derivatives of the cotangent and cosecant functions. We prove it under the completely generic condition, and only use Liouville's theorem and limit of some periodic functions at  $z \rightarrow i\infty$ . Thus, our proof is more generic than the method of Beck, and much simpler than Fukuhara's proof which needs some non-trivial trigonometric identities. Furthermore, from various specializations of our formula, we derive various reciprocity laws of the generalized Dedekind sums uniformly, which include the results in [1] and [5] et al..

Let us now describe the content in this article. In Section 2, we introduce the main object  $\varphi_N^{(I)}(z)$  and  $\varphi_N^{(II)}(z)$  instead of the higher derivatives of the cotangent and cosecant functions, and recall their fundamental properties. In Section 3 which is the main part of this article, under the general situation for the parameters, we provide a product-to-sum type formula for

$$\prod_{l=1}^{j_I} a_l^{m_l} \varphi_{m_l}^{(I)}(a_l z - w_l) \prod_{l=j_I+1}^{j_I+j_{II}} a_l^{m_l} \varphi_{m_l}^{(II)}(a_l z - w_l)$$

and derive new generalized reciprocity laws by writing down some specializations of the main theorem. In Section 4, we restrict parameters of our main results in Section 3 and give more explicit expression of our reciprocity laws. By these specializations, we show that our main results contain a lot of formulas for the generalized Dedekind sums by the distinguished mathematicians. Finally, in Section 5, we present a future work for a variation on a theme of our formulas.

## 2 Preliminaries

Throughout the paper, we denote the ring of rational integers by  $\mathbb{Z}$ , the field of real numbers by  $\mathbb{R}$ , the field of complex numbers by  $\mathbb{C}$  and  $i := \sqrt{-1}$ . Further we use the notation:

$$\mathfrak{R} := \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z < 1\}.$$

First, from Walker's book [7], we recall the two kinds of the periodic functions which play central roles in this article. For a positive integer  $N$ , we define the periodic functions by

$$\varphi_N^{(J)}(z) := \frac{1}{z^N} + \sum_{n=1}^{\infty} (-1)^{n\delta_{J,II}} \left( \frac{1}{(z+n)^N} + \frac{1}{(z-n)^N} \right) \quad (J = I, II). \quad (2.1)$$

In [7],  $\varphi_N^{(I)}(z)$  and  $\varphi_N^{(II)}(z)$  are denoted by  $E_N(z)$  and  $G_N(z)$  respectively. In the following, we list the main properties of  $\varphi_N^{(J)}(z)$  ( $J = I, II$ ) from [7].

Periodicity For any  $\mu \in \mathbb{Z}$ ,

$$\varphi_N^{(J)}(z + \mu) = (-1)^{\mu\delta_{J,II}} \varphi_N^{(J)}(z). \quad (2.2)$$

Derivation For any  $N \in \mathbb{Z}_{\geq 0}$ ,

$$\varphi_{N+1}^{(J)}(z) = \frac{(-1)^N}{N!} \left( \frac{d}{dz} \right)^N \varphi_1^{(J)}(z). \quad (2.3)$$

In particular,

$$\frac{d\varphi_N^{(J)}}{dz}(z) = -N\varphi_{N+1}^{(J)}(z). \quad (2.4)$$

Laurent expansions Let  $\zeta(s)$  be the Riemann zeta function and

$$\alpha_\mu^{(I)} := \begin{cases} 2\zeta(\mu) = (-1)^{\frac{\mu}{2}+1} \frac{B_\mu}{\mu!} (2\pi)^\mu & (\text{if } \mu \text{ is even}) \\ 0 & (\text{if } \mu \text{ is odd}) \end{cases}, \quad (2.5)$$

$$\alpha_\mu^{(II)} := \begin{cases} 2(1 - 2^{1-\mu})\zeta(\mu) = 2(2^{\mu-1} - 1)(-1)^{\frac{\mu}{2}+1} \frac{B_\mu}{\mu!} \pi^\mu & (\text{if } \mu \text{ is even}) \\ 0 & (\text{if } \mu \text{ is odd}) \end{cases}. \quad (2.6)$$

Then, around  $z = 0$ , we have

$$\varphi_N^{(J)}(z) = \frac{1}{z^N} + (-1)^N \sum_{\nu \geq 0} \binom{N + \nu - 1}{N - 1} \alpha_{N+\nu}^{(J)} z^\nu, \quad (2.7)$$

where  $\binom{N+\nu-1}{N-1}$  is the binomial coefficient. More generally, the following result holds. For  $X \subset \mathbb{C}$ , we put

$$\delta_X(z) := \begin{cases} 1 & (\text{if } z \in X) \\ 0 & (\text{if } z \notin X) \end{cases}.$$

and define the signature by

$$\text{sgn}^{(J)}(z_0; a, w) := (-1)^{(az_0 - w)\delta_{J,II}} \delta_{\mathbb{Z}}(az_0 - w). \quad (2.8)$$

By the periodicity (2.2), if  $\delta_{\mathbb{Z}}(az_0 - w) = 1$ , then

$$\varphi_N^{(J)}(z - (az_0 - w)) = \text{sgn}^{(J)}(z_0; a, w) \varphi_N^{(J)}(z).$$

**Lemma 2.1.** *For any  $a, m \in \mathbb{Z}_{\geq 1}$  and  $w, z_0 \in \mathbb{C}$ , we have*

$$a^m \varphi_m^{(J)}(az - w) = \text{sgn}^{(J)}(z_0; a, w) (z - z_0)^{-m} + \sum_{\nu \geq 0} A_\nu^{(J)}(z_0; a, m, w) (z - z_0)^\nu. \quad (2.9)$$

Here,

$$\begin{aligned} (m)_\nu &:= \begin{cases} m(m+1) \cdots (m+\nu-1) & (\text{if } \nu \geq 1) \\ 1 & (\text{if } \nu = 0) \end{cases} \\ A_\nu^{(J)}(z_0; a, m, w) &:= (-1)^m \text{sgn}^{(J)}(z_0; a, w) \binom{m+\nu-1}{m-1} \alpha_{m+\nu}^{(J)} a^{m+\nu} \delta_{\mathbb{Z}}(az_0 - w) \\ &\quad + (-1)^\nu \frac{(m)_\nu}{\nu!} a^{m+\nu} \text{Res}_{z=z_0} \frac{\varphi_{m+\nu}^{(J)}(az - w)}{z - z_0} dz (1 - \delta_{\mathbb{Z}}(az_0 - w)) \\ &= \begin{cases} (-1)^m \text{sgn}^{(J)}(z_0; a, w) \binom{m+\nu-1}{m-1} \alpha_{m+\nu}^{(J)} a^{m+\nu} & (\text{if } \delta_{\mathbb{Z}}(az_0 - w) = 1) \\ (-1)^\nu \frac{(m)_\nu}{\nu!} \varphi_{m+\nu}^{(J)}(az_0 - w) a^{m+\nu} & (\text{if } \delta_{\mathbb{Z}}(az_0 - w) = 0) \end{cases}. \end{aligned}$$

*Proof.* If  $az_0 - w \in \mathbb{C} \setminus \mathbb{Z}$ , that means  $\delta_{\mathbb{Z}}(az_0 - w) = 0$ , then  $z_0$  is not a pole of  $a^m \varphi_m^{(J)}(az - w)$  and

$$\left( \frac{d}{dz} \right)^\nu a^m \varphi_m^{(J)}(az - w) \Big|_{z=z_0} = (-1)^\nu (m)_\nu \varphi_{m+\nu}^{(J)}(az_0 - w) a^{m+\nu}.$$

Thus, from the Taylor expansion of  $a^m \varphi_m^{(J)}(az - w)$  at  $z = z_0$ , we have

$$a^m \varphi_m^{(J)}(az - w) = \sum_{\nu \geq 0} (-1)^\nu \frac{(m)_\nu}{\nu!} \varphi_{m+\nu}^{(J)}(az_0 - w) a^{m+\nu} (z - z_0)^\nu.$$

If  $az_0 - w \in \mathbb{Z}$ , that is  $\delta_{\mathbb{Z}}(az_0 - w) = 1$  case, then there exists  $\mu \in \mathbb{Z}$  such that

$$z_0 = \frac{w + \mu}{a}.$$

Hence, by using the periodicity (2.2) and the Laurent expansion at  $z_0 = 0$  (2.7), we have

$$\begin{aligned} a^m \varphi_m^{(J)}(az - w) &= \operatorname{sgn}^{(J)}(z_0; a, w) a^m \varphi_m^{(J)}(az - w - (az_0 - w)) \\ &= \operatorname{sgn}^{(J)}(z_0; a, w) a^m \varphi_m^{(J)}(a(z - z_0)) \\ &= \operatorname{sgn}^{(J)}(z_0; a, w) (z - z_0)^{-m} \\ &\quad + (-1)^m \operatorname{sgn}^{(J)}(z_0; a, w) \sum_{\nu \geq 0} \binom{m + \nu - 1}{m - 1} \alpha_{m+\nu}^{(J)} a^{m+\nu} (z - z_0)^\nu. \end{aligned}$$

□

### Relationship with the cotangent and cosecant functions

$$\varphi_1^{(I)}(z) = \pi \cot(\pi z), \quad \varphi_1^{(II)}(z) = \pi \csc(\pi z). \quad (2.10)$$

### Limit at $z \rightarrow i\infty$

**Lemma 2.2.**

$$\lim_{z \rightarrow i\infty} \varphi_N^{(J)}(z) = -\pi i \delta_{N,1} \delta_{J,I}. \quad (2.11)$$

*Proof.* For  $N \geq 2$ , since  $\varphi_N^{(J)}(z)$  is absolutely convergent,  $\lim_{z \rightarrow i\infty} \varphi_N^{(J)}(z) = 0$ . If  $N = 1$ , then

$$\begin{aligned} \lim_{z \rightarrow i\infty} \varphi_1^{(I)}(z) &= \lim_{z \rightarrow i\infty} \pi \cot(\pi z) = -\pi i, \\ \lim_{z \rightarrow i\infty} \varphi_1^{(II)}(z) &= \lim_{z \rightarrow i\infty} \pi \csc(\pi z) = 0. \end{aligned}$$

□

## 3 Main results

Let  $r \in \mathbb{Z}_{\geq 2}$ ,  $[r] := \{1, \dots, r\}$ ,  $\mathbf{a} := (a_1, \dots, a_r)$ ,  $\mathbf{m} := (m_1, \dots, m_r) \in \mathbb{Z}_{\geq 1}^r$ ,  $\mathbf{w} = (w_1, \dots, w_r) \in \mathfrak{R}^r$ ,  $\mathbf{j} = (j_I, j_{II}) \in \mathbb{Z}_{\geq 0}^2$ . Here, suppose that  $j_I + j_{II} = r$ . Further, we put

$$\begin{aligned} R_\rho &:= (R_\rho(\mathbf{a}, \mathbf{w}) =) \{\Lambda \subset [r] \mid \delta_{\mathbb{Z}}(a_\lambda \rho - w_\lambda) = 1 \text{ (for all } \lambda \in \Lambda)\}, \\ \Lambda^c &:= [r] \setminus \Lambda, \end{aligned}$$

$$K_{n,\Lambda}^\pm := \left\{ (\nu_k)_{k \in \Lambda^c} \in \mathbb{Z}_{\geq 0}^{|\Lambda^c|} \mid n = \pm \left( \sum_{k \in \Lambda^c} \nu_k - \sum_{\lambda \in \Lambda} m_\lambda \right) \right\},$$

$$\delta_I^{(\mathbf{j}, l)} := \sum_{j=1}^{j_I} \delta_{j,l} = \begin{cases} 1 & \text{(if } 1 \leq l \leq j_I) \\ 0 & \text{(otherwise)} \end{cases}, \quad \delta_{II}^{(\mathbf{j}, l)} := \sum_{j=j_I+1}^{j_I+j_{II}} \delta_{j,l} = \begin{cases} 1 & \text{(if } j_I+1 \leq l \leq j_I+j_{II}) \\ 0 & \text{(otherwise)} \end{cases},$$

and

$$\begin{aligned}
\operatorname{sgn}^{(\mathbf{j},l)}(z_0; a, w) &:= \operatorname{sgn}^{(I)}(z_0; a, w) \delta_I^{(\mathbf{j},l)} + \operatorname{sgn}^{(II)}(z_0; a, w) \delta_{II}^{(\mathbf{j},l)} \\
&= (-1)^{(az_0-w)\delta_{II}^{(\mathbf{j},l)}} \delta_{\mathbb{Z}}(az_0 - w), \\
\varphi_N^{(\mathbf{j},l)}(z) &:= \varphi_N^{(I)}(z) \delta_I^{(\mathbf{j},l)} + \varphi_N^{(II)}(z) \delta_{II}^{(\mathbf{j},l)}, \\
\alpha_\nu^{(\mathbf{j},l)} &:= \alpha_\nu^{(I)} \delta_I^{(\mathbf{j},l)} + \alpha_\nu^{(II)} \delta_{II}^{(\mathbf{j},l)}, \\
A_\nu^{(\mathbf{j},l)}(z_0; a, m, w) &:= A_\nu^{(I)}(z_0; a, m, w) \delta_I^{(\mathbf{j},l)} + A_\nu^{(II)}(z_0; a, m, w) \delta_{II}^{(\mathbf{j},l)}.
\end{aligned}$$

Moreover, for convenience, we consider the following two cases according to  $\mathbf{a}$  and  $\mathbf{j}$ .

$$\text{Case. } I : j_{II} = 0, \text{ or } \sum_{l=j_I+1}^r a_l \text{ is even.}$$

$$\text{Case. } II : \sum_{l=j_I+1}^r a_l \text{ is odd.}$$

The following theorem is the main result of this article.

**Theorem 3.1.** *Let*

$$\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) := \prod_{l=1}^r a_l^{m_l} \varphi_{m_l}^{(\mathbf{j},l)}(a_l z - w_l).$$

Here, if a  $j_J$  is zero, we omit the product term for  $\varphi_{m_l}^{(J)}(a_l z - w_l)$ . For Case.  $J$ , we have

$$\begin{aligned}
\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) &= \cos\left(\frac{\pi r}{2}\right) \pi^r \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1} \\
&\quad + \sum_{n=1}^{|\mathbf{m}|} \sum_{\rho} \sum_{\Lambda \in R_{\rho}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{n,\Lambda}^-} \prod_{l \in \Lambda} (-1)^{(a_l \rho - w_l) \delta_{II}^{(\mathbf{j},l)}} \\
&\quad \cdot \prod_{u \in \Lambda^c} \{A_{\nu_u}^{(\mathbf{j},u)}(\rho; a_u, m_u, w_u)\} \varphi_n^{(J)}(z - \rho), \tag{3.1}
\end{aligned}$$

where  $\rho$  runs over all poles of  $\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w})$  in  $\Re$ , and

$$|\mathbf{m}| := m_1 + \cdots + m_r.$$

*Proof.* We denote  $\Psi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w})$  by the right hand side of (3.1). We claim that for all Case.  $I, II$ ,

$$\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) - \Psi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) = 0.$$

First, we consider the Laurent expansion of  $\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w})$  at  $\rho$

$$\begin{aligned}
\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) &= \prod_{l=1}^r \left\{ \operatorname{sgn}^{(\mathbf{j}, l)}(\rho; a_l, w_l)(z - \rho)^{-m_l} + \sum_{\nu_l \geq 0} A_{\nu}^{(\mathbf{j}, l)}(\rho; a_l, m_l, w_l)(z - \rho)^{\nu_l} \right\} \\
&= \prod_{l=1}^r \{ \operatorname{sgn}^{(\mathbf{j}, l)}(\rho; a_l, w_l) \} (z - \rho)^{-|\mathbf{m}|} + \sum_{N=1}^{r-1} \sum_{1 \leq \lambda_1 < \dots < \lambda_N \leq r} \\
&\quad \cdot \left\{ \sum_{\substack{|\mathbf{m}| > \sum_{k=1}^N (m_{\lambda_k} + \nu_{\lambda_k}), \\ \nu_{\lambda_1}, \dots, \nu_{\lambda_N} \geq 0}} + \sum_{\substack{|\mathbf{m}| \leq \sum_{k=1}^N (m_{\lambda_k} + \nu_{\lambda_k}), \\ \nu_{\lambda_1}, \dots, \nu_{\lambda_N} \geq 0}} \right\} \prod_{l \in [r] \setminus \{\lambda_1, \dots, \lambda_N\}} \operatorname{sgn}^{(\mathbf{j}, l)}(\rho; a_l, w_l) \\
&\quad \cdot \prod_{u=1}^N A_{\nu_{\lambda_u}}^{(\mathbf{j}, \lambda_u)}(\rho; a_{\lambda_u}, m_{\lambda_u}, w_{\lambda_u})(z - \alpha_j)^{-|\mathbf{m}| + \sum_{k=1}^N (m_{\lambda_k} + \nu_{\lambda_k})} \\
&= \sum_{n=1}^{|\mathbf{m}|} \sum_{\Lambda \in R_\rho} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{n, \Lambda}^-} \prod_{l \in \Lambda} (-1)^{(a_l \rho - w_l) \delta_H^{(\mathbf{j}, l)}} \prod_{u \in \Lambda^c} \{ A_{\nu_u}^{(\mathbf{j}, u)}(\rho; a_u, m_u, w_u) \} (z - \rho)^{-n} \\
&\quad + \sum_{\mu \geq 0} \sum_{\Lambda \in R_\rho} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{\mu, \Lambda}^+} \prod_{l \in \Lambda} (-1)^{(a_l \rho - w_l) \delta_H^{(\mathbf{j}, l)}} \\
&\quad \cdot \prod_{u \in \Lambda^c} \{ A_{\nu_u}^{(\mathbf{j}, u)}(\rho; a_u, m_u, w_u) \} (z - \rho)^\mu. \tag{3.2}
\end{aligned}$$

Further, from (2.2), we have

$$\Phi^{(\mathbf{j})}(z + \mu; \mathbf{a}, \mathbf{m}, \mathbf{w}) = (-1)^{\mu \delta_{J, H}} \Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}). \tag{3.3}$$

Thus, for Case. *I, II*,  $\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w})$  and  $\Phi_N^{(J)}(z)$  are the periodic functions with same period. In addition, by (2.9) and (3.2),  $\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) - \Psi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w})$  is entire.

Next, we remark that  $\varphi_N^{(J)}(z)$  is bounded on the set  $\mathfrak{R}_1 := \mathfrak{R} \cap \{z \in \mathbb{C} \mid |\operatorname{Im} z| \geq 1 + 2 \max_\rho |\operatorname{Im} \rho|\}$  from (2.11). Thus,  $\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) - \Psi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w})$  is also bounded on  $\mathfrak{R}_1$ . Hence, by the periodicity,  $\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) - \Psi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w})$  is bounded on  $\mathbb{C}$ . Then by the well-known Liouville's theorem, there exists a constant  $c^{(\mathbf{j}, J)}(\mathbf{a}, \mathbf{m}, \mathbf{w})$  such that

$$\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) - \Psi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) = c^{(\mathbf{j}, J)}(\mathbf{a}, \mathbf{m}, \mathbf{w}).$$

If we restrict  $z \in \mathbb{C}$  and  $w_1, \dots, w_r \in \mathfrak{R}$  to  $\mathbb{R}$ , then  $A_{\nu_u}^{(\mathbf{j}, u)}(\rho; a_u, m_u, w_u), c^{(\mathbf{j}, J)}(\mathbf{a}, \mathbf{m}, \mathbf{w}) \in \mathbb{R}$ .



In addition, we calculate

$$\begin{aligned}
\lim_{z \rightarrow i\infty} \Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) &= \prod_{l=1}^r a_l^{m_l} (-\pi i) \delta_{m_l,1} \delta_I^{(\mathbf{j},l)} \\
&= (-i)^r \pi^r \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1} \\
&= \left\{ \cos\left(\frac{\pi r}{2}\right) - i \sin\left(\frac{\pi r}{2}\right) \right\} \pi^r \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1}, \tag{3.4}
\end{aligned}$$

$$\begin{aligned}
\lim_{z \rightarrow i\infty} \Psi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) &= \cos\left(\frac{\pi r}{2}\right) \pi^r \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1} \\
&\quad - \pi i \delta_{J,I} \sum_{\rho} \sum_{\Lambda \in R_{\rho}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{1,\Lambda}^-} \prod_{l \in \Lambda} (-1)^{(a_l \rho - w_l) \delta_{II}^{(\mathbf{j},l)}} \\
&\quad \cdot \prod_{u \in \Lambda^c} A_{\nu_u}^{(\mathbf{j},u)}(\rho; a_u, m_u, w_u). \tag{3.5}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
c^{(\mathbf{j},J)}(\mathbf{a}, \mathbf{m}, \mathbf{w}) &= \operatorname{Re} \{ c^{(\mathbf{j},J)}(\mathbf{a}, \mathbf{m}, \mathbf{w}) \} \\
&= \operatorname{Re} \left\{ \lim_{z \rightarrow i\infty} \{ \Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) - \Psi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) \} \right\} = 0.
\end{aligned}$$

□

As a corollary of this theorem, we obtain the following theorem immediately.

**Theorem 3.2.** (1) *We have*

$$\begin{aligned}
&\sum_{\rho} \sum_{\Lambda \in R_{\rho}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{1,\Lambda}^-} \prod_{l \in \Lambda} (-1)^{(a_l \rho - w_l) \delta_{II}^{(\mathbf{j},l)}} \prod_{u \in \Lambda^c} A_{\nu_u}^{(\mathbf{j},u)}(\rho; a_u, m_u, w_u) \\
&= \pi^{r-1} \sin\left(\frac{\pi r}{2}\right) \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1}. \tag{3.6}
\end{aligned}$$

(2) *For any  $z_0 \in \mathbb{C}$ ,  $\mu \in \mathbb{Z}_{\geq 0}$ , we have*

$$\begin{aligned}
&\sum_{n=1}^{|\mathbf{m}|} \sum_{\rho} \sum_{\Lambda \in R_{\rho}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{n,\Lambda}^-} \prod_{l \in \Lambda} (-1)^{(a_l \rho - w_l) \delta_{II}^{(\mathbf{j},l)}} \prod_{u \in \Lambda^c} \{ A_{\nu_u}^{(\mathbf{j},u)}(\rho; a_u, m_u, w_u) \} A_{\mu}^{(J)}(z_0; 1, n, \rho) \\
&= -\cos\left(\frac{\pi r}{2}\right) \pi^r \delta_{\mu,0} \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1} \\
&\quad + \sum_{\Lambda \in R_{z_0}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{\mu,\Lambda}^+} \prod_{l \in \Lambda} (-1)^{(a_l z_0 - w_l) \delta_{II}^{(\mathbf{j},l)}} \prod_{u \in \Lambda^c} A_{\nu_u}^{(\mathbf{j},u)}(z_0; a_u, m_u, w_u). \tag{3.7}
\end{aligned}$$

*Proof.* (1) It follows from (3.4) and (3.5) immediately.

(2) We expand both sides of (3.1) into the Laurent series of  $z - z_0$  and compare the coefficients of  $(z - z_0)^\mu$  of both sides. Indeed, by replacing  $\rho$  with  $z_0$  in (3.2), we have

$$\operatorname{Res}_{z=z_0} \frac{\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w})}{(z - z_0)^{\mu+1}} dz = \sum_{\Lambda \in R_{z_0}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{\mu, \Lambda}^+} \prod_{l \in \Lambda} (-1)^{(a_l z_0 - w_l) \delta_{II}^{(\mathbf{j}, l)}} \prod_{u \in \Lambda^c} A_{\nu_u}^{(\mathbf{j}, u)}(z_0; a_u, m_u, w_u).$$

On the other hand, from (3.1),

$$\begin{aligned} \operatorname{Res}_{z=z_0} \frac{\Psi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w})}{(z - z_0)^{\mu+1}} dz &= \cos\left(\frac{\pi r}{2}\right) \pi^r \delta_{\mu, 0} \delta_{j_{II}, 0} \prod_{l=1}^r a_l \delta_{m_l, 1} \\ &\quad + \sum_{n=1}^{|\mathbf{m}|} \sum_{\rho} \sum_{\Lambda \in R_{\rho}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{n, \Lambda}^-} \prod_{l \in \Lambda} (-1)^{(a_l \rho - w_l) \delta_{II}^{(\mathbf{j}, l)}} \\ &\quad \cdot \prod_{u \in \Lambda^c} \{A_{\nu_u}^{(\mathbf{j}, u)}(\rho; a_u, m_u, w_u)\} A_{\mu}^{(J)}(z_0; 1, n, \rho). \end{aligned}$$

Therefore, we have the conclusion.  $\square$

As we will see later, Theorem 3.2 (1) is a generalization of various reciprocity laws in [1]. Theorem 3.2 (2) means (3.1) is regarded as a generating function of the reciprocity laws (3.7). Hence, this result and proof are generalizations of theorem 1.2 in [4] and its proof.

**Remark 3.3.** For Theorems 3.1, 3.2, we also have the other expressions which are useful for writing down various specific examples. Let

$$d_v^{(\mu)} := \# \left\{ a_j \in \{a_1, \dots, a_r\}, w_j \in \{w_1, \dots, w_r\}, \mu_j \in \{0, \dots, a_j - 1\} \left| \frac{w_v + \mu}{a_v} = \frac{w_j + \mu_j}{a_j} \right. \right\}.$$

Our main result (3.1) becomes

$$\begin{aligned} \Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) &= \cos\left(\frac{\pi r}{2}\right) \pi^r \delta_{j_{II}, 0} \prod_{l=1}^r a_l \delta_{m_l, 1} \\ &\quad + \sum_{n=1}^{|\mathbf{m}|} \sum_{v=1}^r \sum_{\mu_v=0}^{a_v-1} \frac{1}{d_v^{(\mu_v)}} \sum_{\Lambda \in R_{\frac{w_v + \mu_v}{a_v}}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{n, \Lambda}^-} \prod_{l \in \Lambda} (-1)^{(a_l \frac{w_v + \mu_v}{a_v} - w_l) \delta_{II}^{(\mathbf{j}, l)}} \\ &\quad \cdot \prod_{u \in \Lambda^c} \left\{ A_{\nu_u}^{(\mathbf{j}, u)} \left( \frac{w_v + \mu_v}{a_v}; a_u, m_u, w_u \right) \right\} \varphi_n^{(J)} \left( z - \frac{w_v + \mu_v}{a_v} \right). \end{aligned} \quad (3.8)$$

Similarly, (3.6) and (3.7) are

$$\begin{aligned} &\sum_{v=1}^r \sum_{\mu_v=0}^{a_v-1} \frac{1}{d_v^{(\mu_v)}} \sum_{\Lambda \in R_{\frac{w_v + \mu_v}{a_v}}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{1, \Lambda}^-} \prod_{l \in \Lambda} (-1)^{(a_l \frac{w_v + \mu_v}{a_v} - w_l) \delta_{II}^{(\mathbf{j}, l)}} \prod_{u \in \Lambda^c} A_{\nu_u}^{(\mathbf{j}, u)} \left( \frac{w_v + \mu_v}{a_v}; a_u, m_u, w_u \right) \\ &= \pi^{r-1} \sin\left(\frac{\pi r}{2}\right) \delta_{j_{II}, 0} \prod_{l=1}^r a_l \delta_{m_l, 1}. \end{aligned} \quad (3.9)$$

and for any  $z_0 \in \mathbb{C}, \mu \in \mathbb{Z}_{\geq 0}$ ,

$$\begin{aligned}
& \sum_{n=1}^{|\mathbf{m}|} \sum_{v=1}^r \sum_{\mu_v=0}^{a_v-1} \frac{1}{d_v^{(\mu_v)}} \sum_{\Lambda \in R_{\frac{w_v+\mu_v}{a_v}}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{n,\Lambda}^-} \prod_{l \in \Lambda} (-1)^{(a_l \frac{w_v+\mu_v}{a_v} - w_l) \delta_{II}^{(j,l)}} \\
& \prod_{u \in \Lambda^c} \left\{ A_{\nu_u}^{(j,u)} \left( \frac{w_v + \mu_v}{a_v}; a_u, m_u, w_u \right) \right\} A_{\mu}^{(J)} \left( z_0; 1, n, \frac{w_v + \mu_v}{a_v} \right) \\
& = -\cos\left(\frac{\pi r}{2}\right) \pi^r \delta_{\mu,0} \delta_{jII,0} \prod_{l=1}^r a_l \delta_{m_l,1} \\
& + \sum_{\Lambda \in R_{z_0}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{\mu,\Lambda}^+} \prod_{l \in \Lambda} (-1)^{(a_l z_0 - w_l) \delta_{II}^{(j,l)}} \prod_{u \in \Lambda^c} A_{\nu_u}^{(j,u)}(z_0; a_u, m_u, w_u). \tag{3.10}
\end{aligned}$$

respectively.

## 4 Some special cases of the main theorem

By specializing our main results, we derive various reciprocity laws of the generalized Dedekind sums.

### 4.1 Multiplicity free case

In this subsection, we assume for all distinct  $k, l \in [r]$  and  $\mu_k = 0, 1, \dots, a_k - 1$ ,  $\mu_l = 0, 1, \dots, a_l - 1$ ,

$$\frac{w_k + \mu_k}{a_k} \neq \frac{w_l + \mu_l}{a_l}.$$

Under this condition, all poles of  $a_j^{m_j} \varphi_{m_j}^{(J)}(a_j z - w_j)$  for each  $j = 1, \dots, r$  on  $\Re$

$$\frac{w_j + \mu_j}{a_j} \quad (j = 1, \dots, r, \text{ and } \mu_j = 0, \dots, a_j - 1)$$

are multiplicity free, that means

$$\delta_{\mathbb{Z}} \left( a_l \frac{w_j + \mu_j}{a_j} - w_l \right) = \begin{cases} 1 & (\text{if } l = j) \\ 0 & (\text{if } l \neq j) \end{cases},$$

and  $d_v^{(\mu_v)} = 1$  for all  $1 \leq v \leq r$ ,  $0 \leq \mu_v \leq a_v - 1$ . Hence, Theorems 3.1, 3.2 are as follows.

**Theorem 4.1.** *We have*

$$\begin{aligned}
\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{w}) &= \cos\left(\frac{\pi r}{2}\right) \pi^r \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1} \\
&+ \sum_{n=1}^{\max_{j \in [r]} \{m_j\}} \sum_{l=1}^r \sum_{\mu_l=0}^{a_l-1} \sum_{\substack{n=m_l - \sum_{1 \leq k \neq l \leq r} \nu_k, \\ \nu_1, \dots, \nu_r \geq 0}} (-1)^{\mu_l \delta_{II}^{(\mathbf{j},l)}} \\
&\cdot \prod_{1 \leq u \neq l \leq r} \left\{ (-1)^{\nu_u} \frac{(m_u)_{\nu_u}}{\nu_u!} \varphi_{m_u+\nu_u}^{(\mathbf{j},u)} \left( a_u \frac{w_l + \mu_l}{a_l} - w_u \right) a_u^{m_u+\nu_u} \right\} \\
&\cdot \varphi_n^{(J)} \left( z - \frac{w_l + \mu_l}{a_l} \right). \tag{4.1}
\end{aligned}$$

**Theorem 4.2.** (1) *We have*

$$\begin{aligned}
&\sum_{l=1}^r \sum_{\mu_l=0}^{a_l-1} \sum_{\substack{n=m_l - \sum_{1 \leq k \neq l \leq r} \nu_k, \\ \nu_1, \dots, \nu_r \geq 0}} (-1)^{\mu_l \delta_{II}^{(\mathbf{j},l)}} \\
&\cdot \prod_{1 \leq u \neq l \leq r} \left\{ (-1)^{\nu_u} \frac{(m_u)_{\nu_u}}{\nu_u!} \varphi_{m_u+\nu_u}^{(\mathbf{j},u)} \left( a_u \frac{w_l + \mu_l}{a_l} - w_u \right) a_u^{m_u+\nu_u} \right\} = \pi^{r-1} \sin\left(\frac{\pi r}{2}\right) \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1}. \tag{4.2}
\end{aligned}$$

*In particular, for  $m_1 = \dots = m_r = 1$ ,*

$$\sum_{l=1}^r \sum_{\mu_l=0}^{a_l-1} (-1)^{\mu_l \delta_{II}^{(\mathbf{j},l)}} \prod_{1 \leq u \neq l \leq r} \left\{ \varphi_1^{(\mathbf{j},u)} \left( a_u \frac{w_l + \mu_l}{a_l} - w_u \right) a_u \right\} = \pi^{r-1} \sin\left(\frac{\pi r}{2}\right) \delta_{j_{II},0} \prod_{l=1}^r a_l. \tag{4.3}$$

(2) *For any  $\mu \in \mathbb{Z}_{\geq 0}$  and  $z_0 \in \mathfrak{R}$ ,*

$$\begin{aligned}
&\sum_{n=1}^{\max_{j \in [r]} \{m_j\}} \sum_{l=1}^r \sum_{\mu_l=0}^{a_l-1} \sum_{\substack{n=m_l - \sum_{1 \leq k \neq l \leq r} \nu_k, \\ \nu_1, \dots, \nu_r \geq 0}} (-1)^{\mu_l \delta_{II}^{(\mathbf{j},l)}} \\
&\cdot \prod_{1 \leq u \neq l \leq r} \left\{ (-1)^{\nu_u} \frac{(m_u)_{\nu_u}}{\nu_u!} \varphi_{m_u+\nu_u}^{(\mathbf{j},u)} \left( a_u \frac{w_l + \mu_l}{a_l} - w_u \right) a_u^{m_u+\nu_u} \right\} A_\mu^{(J)} \left( z_0; 1, n, \frac{w_l + \mu_l}{a_l} \right) \\
&= -\cos\left(\frac{\pi r}{2}\right) \pi^r \delta_{\mu,0} \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1} \\
&+ \sum_{\Lambda \in R_{z_0}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{\mu, \Lambda}^+} \prod_{l \in \Lambda} (-1)^{(a_l z_0 - w_l) \delta_{II}^{(\mathbf{j},l)}} \prod_{u \in \Lambda^c} A_{\nu_u}^{(\mathbf{j},u)}(z_0; a_u, m_u, w_u). \tag{4.4}
\end{aligned}$$

**Example 4.3.** When we consider the case of  $(j_I, j_{II}) = (r, 0)$ , (4.2) is none other than Beck's reciprocity (Theorem 2 in [1])

$$\begin{aligned} & \sum_{l=1}^r \sum_{\mu_l=0}^{a_l-1} \sum_{1=m_l-\sum_{1 \leq k \neq l \leq r} \nu_k, 1 \leq u \neq l \leq r} \prod_{\nu_1, \dots, \nu_r \geq 0} \left\{ (-1)^{\nu_u} \frac{(m_u)_{\nu_u}}{\nu_u!} \varphi_{m_u+\nu_u}^{(I)} \left( a_u \frac{w_l + \mu_l}{a_l} - w_u \right) a_u^{m_u+\nu_u} \right\} \\ &= \pi^{r-1} \sin\left(\frac{\pi r}{2}\right) \prod_{l=1}^r a_l \delta_{m_l, 1}. \end{aligned} \quad (4.5)$$

On the other hand, by putting  $(j_I, j_{II}) = (0, r)$ , we obtain a cosecant analogue of Beck's result

$$\sum_{l=1}^r \sum_{\mu_l=0}^{a_l-1} \sum_{1=m_l-\sum_{1 \leq k \neq l \leq r} \nu_k, 1 \leq u \neq l \leq r} (-1)^{\mu_l} \prod_{\nu_1, \dots, \nu_r \geq 0} \left\{ (-1)^{\nu_u} \frac{(m_u)_{\nu_u}}{\nu_u!} \varphi_{m_u+\nu_u}^{(II)} \left( a_u \frac{w_l + \mu_l}{a_l} - w_u \right) a_u^{m_u+\nu_u} \right\} = 0. \quad (4.6)$$

## 4.2 $\mathbf{w} = (0, \dots, 0)$ case

In this subsection, we assume  $\mathbf{w} = \mathbf{0} := (0, \dots, 0)$  and  $a_1, \dots, a_r$  are pairwise relatively prime. Under this condition, all poles of  $a_j^{m_j} \varphi_{m_j}^{(J)}(a_j z)$  for each  $j = 1, \dots, r$  on  $\Re$  are

$$\frac{\mu_j}{a_j} \quad (j = 1, \dots, r, \text{ and } \mu_j = 0, \dots, a_j - 1).$$

Further,  $\delta_{\mathbb{Z}}(0) = 1$  and for all  $l = 1, \dots, r$ , and  $\mu_j = 1, \dots, a_j - 1$ ,

$$\delta_{\mathbb{Z}}\left(a_l \frac{\mu_j}{a_j}\right) = \begin{cases} 1 & (\text{if } l = j) \\ 0 & (\text{if } l \neq j) \end{cases}, \quad d_v^{(\mu_v)} = \begin{cases} r & (\text{if } \mu_v = 0) \\ 1 & (\text{otherwise}) \end{cases}.$$

Therefore, Theorems 3.1, 3.2 degenerate to the following results.

**Theorem 4.4.** *Let*

$$\begin{aligned} M_n^{(\mathbf{j})}(\mathbf{a}, \mathbf{m}) &:= \sum_{\Lambda \in R_0} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{n, \Lambda}^-} \prod_{u \in \Lambda^c} (-1)^{m_u} \binom{m_u + \nu_u - 1}{m_u - 1} \alpha_{m_u+\nu_u}^{(\mathbf{j}, u)} a_u^{m_u+\nu_u} \\ &= \sum_{N=1}^{r-1} \sum_{1 \leq \lambda_1 < \dots < \lambda_N \leq r} \sum_{n=|\mathbf{m}| - \sum_{k=1}^N (\nu_{\lambda_k} + m_{\lambda_k}), \nu_{\lambda_1}, \dots, \nu_{\lambda_N} \geq 0} \prod_{u=1}^N (-1)^{m_u} \binom{m_u + \nu_u - 1}{m_u - 1} \alpha_{m_u+\nu_u}^{(\mathbf{j}, u)} a_u^{m_u+\nu_u}. \end{aligned}$$

We obtain

$$\begin{aligned}
\Phi^{(\mathbf{j})}(z; \mathbf{a}, \mathbf{m}, \mathbf{0}) &= \cos\left(\frac{\pi r}{2}\right) \pi^r \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1} + \sum_{n=1}^{|\mathbf{m}|} M_n^{(\mathbf{j})}(\mathbf{a}, \mathbf{m}) \varphi_n^{(J)}(z) \\
&+ \sum_{n=1}^{\max_{j \in [r]} \{m_j\}} \sum_{l=1}^r \sum_{\mu_l=1}^{a_l-1} \sum_{\substack{n=m_l - \sum_{1 \leq k \neq l \leq r} \nu_k, \\ \nu_1, \dots, \nu_r \geq 0}} (-1)^{\mu_l \delta_{II}^{(\mathbf{j}, l)}} \\
&\cdot \prod_{1 \leq u \neq l \leq r} \left\{ (-1)^{\nu_u} \frac{(m_u)_{\nu_u}}{\nu_u!} \varphi_{m_u + \nu_u}^{(\mathbf{j}, u)} \left( a_u \frac{\mu_l}{a_l} \right) a_u^{m_u + \nu_u} \right\} \varphi_n^{(J)} \left( z - \frac{\mu_l}{a_l} \right). \quad (4.7)
\end{aligned}$$

**Theorem 4.5.** (1) We have

$$\begin{aligned}
&\sum_{l=1}^r \sum_{\mu_l=1}^{a_l-1} \sum_{\substack{n=m_l - \sum_{1 \leq k \neq l \leq r} \nu_k, \\ \nu_1, \dots, \nu_r \geq 0}} (-1)^{\mu_l \delta_{II}^{(\mathbf{j}, l)}} \prod_{1 \leq u \neq l \leq r} \left\{ (-1)^{\nu_u} \frac{(m_u)_{\nu_u}}{\nu_u!} \varphi_{m_u + \nu_u}^{(\mathbf{j}, u)} \left( a_u \frac{\mu_l}{a_l} \right) a_u^{m_u + \nu_u} \right\} \\
&= \pi^{r-1} \sin\left(\frac{\pi r}{2}\right) \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1} - M_1^{(\mathbf{j})}(\mathbf{a}, \mathbf{m}). \quad (4.8)
\end{aligned}$$

In particular, for  $\mathbf{m} = \mathbf{1} := (1, \dots, 1)$ ,

$$\sum_{l=1}^r \sum_{\mu_l=1}^{a_l-1} (-1)^{\mu_l \delta_{II}^{(\mathbf{j}, l)}} \prod_{1 \leq u \neq l \leq r} \left\{ \varphi_1^{(\mathbf{j}, u)} \left( a_u \frac{\mu_l}{a_l} \right) a_u \right\} = \pi^{r-1} \sin\left(\frac{\pi r}{2}\right) \delta_{j_{II},0} \prod_{l=1}^r a_l - M_1^{(\mathbf{j})}(\mathbf{a}, \mathbf{1}). \quad (4.9)$$

(2) For any  $\mu \in \mathbb{Z}_{\geq 0}$  and  $z_0 \in \mathfrak{R}$ ,

$$\begin{aligned}
&\sum_{n=1}^{\max_{j \in [r]} \{m_j\}} \sum_{l=1}^r \sum_{\mu_l=1}^{a_l-1} \sum_{\substack{n=m_l - \sum_{1 \leq k \neq l \leq r} \nu_k, \\ \nu_1, \dots, \nu_r \geq 0}} (-1)^{\mu_l \delta_{II}^{(\mathbf{j}, l)}} \\
&\cdot \prod_{1 \leq u \neq l \leq r} \left\{ (-1)^{\nu_u} \frac{(m_u)_{\nu_u}}{\nu_u!} \varphi_{m_u + \nu_u}^{(\mathbf{j}, u)} \left( a_u \frac{\mu_l}{a_l} \right) a_u^{m_u + \nu_u} \right\} A_\mu^{(J)} \left( z_0; 1, n, \frac{\mu_l}{a_l} \right) \\
&= -\cos\left(\frac{\pi r}{2}\right) \pi^r \delta_{\mu,0} \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1} - \sum_{n=1}^{|\mathbf{m}|} M_n^{(\mathbf{j})}(\mathbf{a}, \mathbf{m}) A_\mu^{(J)}(z_0; 1, n, 0) \\
&+ \sum_{\Lambda \in R_{z_0}} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{\mu, \Lambda}^+} \prod_{l \in \Lambda} (-1)^{a_l z_0 \delta_{II}^{(\mathbf{j}, l)}} \prod_{u \in \Lambda^c} A_{\nu_u}^{(\mathbf{j}, u)}(z_0; a_u, m_u, 0). \quad (4.10)
\end{aligned}$$

In particular, by taking  $z_0 = 0$ ,

$$\begin{aligned}
& \sum_{n=1}^{\max_{j \in [r]} \{m_j\}} \sum_{l=1}^r \sum_{\mu_l=1}^{a_l-1} \sum_{\substack{n=m_l - \sum_{1 \leq k \neq l \leq r} \nu_k \\ \nu_1, \dots, \nu_r \geq 0}} (-1)^{\mu_l \delta_{II}^{(j,l)}} \\
& \cdot \prod_{1 \leq u \neq l \leq r} \left\{ (-1)^{\nu_u} \frac{(m_u)_{\nu_u}}{\nu_u!} \varphi_{m_u+\nu_u}^{(j,u)} \left( a_u \frac{\mu_l}{a_l} \right) a_u^{m_u+\nu_u} \right\} (-1)^\mu \frac{(n)_\mu}{\mu!} \varphi_{n+\mu}^{(J)} \left( -\frac{\mu_l}{a_l} \right) \\
& = -\cos\left(\frac{\pi r}{2}\right) \pi^r \delta_{\mu,0} \delta_{j_{II},0} \prod_{l=1}^r a_l \delta_{m_l,1} - \sum_{n=1}^{|\mathbf{m}|} M_n^{(j)}(\mathbf{a}, \mathbf{m}) (-1)^n \binom{n+\mu-1}{n-1} \alpha_{\mu+n}^{(J)} \\
& + \sum_{\Lambda \in R_0} \sum_{(\nu_k)_{k \in \Lambda^c} \in K_{\mu, \Lambda}^+} \prod_{u \in \Lambda^c} \left\{ (-1)^{m_u} \binom{m_u + \nu_u - 1}{m_u - 1} \alpha_{m_u+\nu_u}^{(j,u)} a_u^{m_u+\nu_u} \right\}. \tag{4.11}
\end{aligned}$$

**Example 4.6.** By putting  $(j_I, j_{II}) = (r, 0)$  in (4.9), we obtain Zagier's result

$$\begin{aligned}
\pi^{r-1} \sum_{l=1}^r \sum_{\mu_l=1}^{a_l-1} \prod_{u \neq l} \left\{ \cot\left(\frac{\pi a_u \mu_l}{a_l}\right) a_u \right\} &= \sin\left(\frac{\pi r}{2}\right) \pi^{r-1} \prod_{l=1}^r a_l \\
&- \sum_{N=1}^{r-1} \sum_{1 \leq \lambda_1 < \dots < \lambda_N \leq r} \sum_{\substack{r-1-N = \sum_{k=1}^N \nu_{\lambda_k}, \\ \nu_{\lambda_1}, \dots, \nu_{\lambda_N} \geq 0}} \prod_{u=1}^N \alpha_{\nu_u+1}^{(I)} a_u^{\nu_u+1}. \tag{4.12}
\end{aligned}$$

Further, if we consider the case of  $(j_I, j_{II}) = (0, r)$ , then

$$\begin{aligned}
\pi^{r-1} \sum_{l=1}^r \sum_{\mu_l=1}^{a_l-1} (-1)^{\mu_l} \prod_{u \neq l} \left\{ \csc\left(\frac{\pi a_u \mu_l}{a_l}\right) a_u \right\} &= - \sum_{N=1}^{r-1} \sum_{1 \leq \lambda_1 < \dots < \lambda_N \leq r} \sum_{\substack{r-1-N = \sum_{k=1}^N \nu_{\lambda_k}, \\ \nu_{\lambda_1}, \dots, \nu_{\lambda_N} \geq 0}} \\
&\cdot \prod_{u=1}^N \alpha_{\nu_u+1}^{(II)} a_u^{\nu_u+1}. \tag{4.13}
\end{aligned}$$

This is a cosecant version of Zagier's reciprocity law.

### 4.3 $r = 2, \mathbf{m} = (1, 1)$ case

In this subsection, we assume  $r = 2, \mathbf{m} = (1, 1)$  and  $a_1, a_2$  are relatively prime. For this simple case, we obtain more explicit expressions for our main results.

**Theorem 4.7.** Let  $I \leq K_1 \leq K_2 \leq II$ , and  $A_1, A_2$  denote integers for which  $A_1 a_2 + A_2 a_1 = 1$  holds. For the following case

$$(K_1, K_2, J) = (I, I, I), (I, II, I), (I, II, II), (II, II, I), (II, II, II), \tag{4.14}$$

we obtain

$$\begin{aligned}
& a_1 a_2 \varphi_1^{(K_1)}(a_1 z - w_1) \varphi_1^{(K_2)}(a_2 z - w_2) \\
&= -\pi^2 a_1 a_2 \delta_{K_1, I} \delta_{K_2, I} \\
&+ \delta_{\mathbb{Z}}(a_1 w_2 - a_2 w_1) \operatorname{sgn}_2^{(K_1, K_2)}((a_1, a_2), (w_1, w_2), (A_1, A_2)) \varphi_2^{(J)}(z - (A_1 w_2 + A_2 w_1)) \\
&+ a_2 \sum_{\mu_1=0}^{a_1-1} '(-1)^{\mu_1 \delta_{K_1, II}} \varphi_1^{(K_2)} \left( a_2 \frac{w_1 + \mu_1}{a_1} - w_2 \right) \varphi_1^{(J)} \left( z - \frac{w_1 + \mu_1}{a_1} \right) \\
&+ a_1 \sum_{\mu_2=0}^{a_2-1} '(-1)^{\mu_2 \delta_{K_2, II}} \varphi_1^{(K_1)} \left( a_1 \frac{w_2 + \mu_2}{a_2} - w_1 \right) \varphi_1^{(J)} \left( z - \frac{w_2 + \mu_2}{a_2} \right). \tag{4.15}
\end{aligned}$$

Here, the sums run over non-singular points and

$$\begin{aligned}
\operatorname{sgn}_2^{(K_1, K_2)}((a_1, a_2), (w_1, w_2), (A_1, A_2)) &:= \operatorname{sgn}^{(K_1)}(A_1 w_2 + A_2 w_1; a_1 + a_2, w_1 + w_2) \delta_{K_1, K_2} \\
&+ \operatorname{sgn}^{(K_1)}(A_1 w_2 + A_2 w_1; a_1, w_1) \\
&\cdot \operatorname{sgn}^{(K_2)}(A_1 w_2 + A_2 w_1; a_2, w_2) (1 - \delta_{K_1, K_2}).
\end{aligned}$$

*Proof.* The multiplicity free case

$$\text{i.e. } \frac{w_1 + \mu_1}{a_1} \neq \frac{w_2 + \mu_2}{a_2} \quad (\mu_1 = 0, 1, \dots, a_1 - 1, \mu_2 = 0, 1, \dots, a_2 - 1)$$

has been proved by some special cases of (4.1). Hence, it is enough to show another case. In the case, since  $a_1, a_2$  are relatively prime and  $w_1, w_2 \in \mathfrak{R}$ , there exists unique integers  $\tilde{\mu}_1 \in \{0, 1, \dots, a_1 - 1\}$  and  $\tilde{\mu}_2 \in \{0, 1, \dots, a_2 - 1\}$  such that

$$\rho_0 := \frac{w_1 + \tilde{\mu}_1}{a_1} = \frac{w_2 + \tilde{\mu}_2}{a_2},$$

and

$$\begin{aligned}
A_1 w_2 + A_2 w_1 &= \rho_0 - (A_1 \tilde{\mu}_2 + A_2 \tilde{\mu}_1), \\
a_1(A_1 w_2 + A_2 w_1) - w_1 &= -a_1(A_1 \tilde{\mu}_2 + A_2 \tilde{\mu}_1) + \tilde{\mu}_1, \\
a_2(A_1 w_2 + A_2 w_1) - w_2 &= -a_2(A_1 \tilde{\mu}_2 + A_2 \tilde{\mu}_1) + \tilde{\mu}_2, \\
(a_1 + a_2)(A_1 w_2 + A_2 w_1) - (w_1 + w_2) &= -(a_1 + a_2)(A_1 \tilde{\mu}_2 + A_2 \tilde{\mu}_1) + (\tilde{\mu}_1 + \tilde{\mu}_2).
\end{aligned}$$

Hence, from (3.1), we have

$$\begin{aligned}
& a_1 a_2 \varphi_1^{(K_1)}(a_1 z - w_1) \varphi_1^{(K_2)}(a_2 z - w_2) \\
&= -\pi^2 a_1 a_2 \delta_{K_1, I} \delta_{K_2, I} + (-1)^{\tilde{\mu}_1 \delta_{II}^{(j,1)} + \tilde{\mu}_2 \delta_{II}^{(j,2)}} \varphi_2^{(J)}(z - \rho_0) \\
&+ a_2 \sum_{\mu_1=0}^{a_1-1} '(-1)^{\mu_1 \delta_{K_1, II}} \varphi_1^{(K_2)} \left( a_2 \frac{w_1 + \mu_1}{a_1} - w_2 \right) \varphi_1^{(J)} \left( z - \frac{w_1 + \mu_1}{a_1} \right) \\
&+ a_1 \sum_{\mu_2=0}^{a_2-1} '(-1)^{\mu_2 \delta_{K_2, II}} \varphi_1^{(K_1)} \left( a_1 \frac{w_2 + \mu_2}{a_2} - w_1 \right) \varphi_1^{(J)} \left( z - \frac{w_2 + \mu_2}{a_2} \right).
\end{aligned}$$



Here, we remark under the above five conditions (4.14)

$$(-1)^{\widetilde{\mu}_1 \delta_{II}^{(j,1)} + \widetilde{\mu}_2 \delta_{II}^{(j,2)}} = (-1)^{\widetilde{\mu}_1 \delta_{K_1, II} \delta_{K_2, II} + \widetilde{\mu}_2 \delta_{K_2, II}}.$$

Thus, by the definition of the signature (2.8) and the periodicity of  $\varphi_N^{(J)}$  (2.2),

$$\begin{aligned} & \text{sgn}_2^{(K_1, K_2)}((a_1, a_2), (w_1, w_2), (A_1, A_2)) \varphi_2^{(J)}(z - (A_1 w_2 + A_2 w_1)) \\ &= \widetilde{\text{sgn}}_2^{((K_1, K_2), J)}((a_1, a_2), (w_1, w_2), (A_1, A_2)) (-1)^{\widetilde{\mu}_1 \delta_{K_1, II} \delta_{K_2, II} + \widetilde{\mu}_2 \delta_{K_2, II}} \varphi_2^{(J)}(z - \rho_0), \end{aligned}$$

where

$$\begin{aligned} & \widetilde{\text{sgn}}_2^{((K_1, K_2), J)}((a_1, a_2), (w_1, w_2), (A_1, A_2)) \\ &:= (-1)^{(A_1 \widetilde{\mu}_2 + A_2 \widetilde{\mu}_1) \{ (a_1 + a_2) \delta_{K_1, II} + \delta_{J, II} \} + \widetilde{\mu}_1 \delta_{K_1, II} (1 + \delta_{K_2, II}) + \widetilde{\mu}_2 (\delta_{K_1, II} + \delta_{K_2, II})} \delta_{K_1, K_2} \\ &+ (-1)^{(A_1 \widetilde{\mu}_2 + A_2 \widetilde{\mu}_1) (a_1 \delta_{K_1, II} + a_2 \delta_{K_2, II} + \delta_{J, II}) + \widetilde{\mu}_1 \delta_{K_1, II} (1 + \delta_{K_2, II})} (1 - \delta_{K_1, K_2}). \end{aligned}$$

Therefore, we claim that for the above five conditions (4.14),

$$\widetilde{\text{sgn}}_2^{((K_1, K_2), J)}((a_1, a_2), (w_1, w_2), (A_1, A_2)) = 1$$

and obtain the conclusion.  $\square$

**Example 4.8.** (0) (Theorem 2.4 in [2])  $(K_1, K_2, J) = (I, I, I)$  case.

$$\begin{aligned} \cot \pi(a_1 z - w_1) \cot \pi(a_2 z - w_2) &= -1 - \frac{1}{a_1 a_2} \delta_{\mathbb{Z}}(a_1 w_2 - a_2 w_1) \cot^{(1)}(\pi(z - (A_1 w_2 + A_2 w_1))) \\ &+ \frac{1}{a_1} \sum_{\mu_1=0}^{a_1-1} \cot \left( \pi \left( a_2 \frac{w_1 + \mu_1}{a_1} - w_2 \right) \right) \cot \left( \pi \left( z - \frac{w_1 + \mu_1}{a_1} \right) \right) \\ &+ \frac{1}{a_2} \sum_{\mu_2=0}^{a_2-1} \cot \left( \pi \left( a_1 \frac{w_2 + \mu_2}{a_2} - w_1 \right) \right) \cot \left( \pi \left( z - \frac{w_2 + \mu_2}{a_2} \right) \right). \end{aligned}$$

(1)  $(K_1, K_2, J) = (I, II, I)$  case.

$$\begin{aligned} & \cot \pi(a_1 z - w_1) \csc \pi(a_2 z - w_2) \\ &= -\frac{(-1)^{a_2(A_1 w_2 + A_2 w_1) - w_2}}{a_1 a_2} \delta_{\mathbb{Z}}(a_1 w_2 - a_2 w_1) \cot^{(1)}(\pi(z - (A_1 w_2 + A_2 w_1))) \\ &+ \frac{1}{a_1} \sum_{\mu_1=0}^{a_1-1} \csc \left( \pi \left( a_2 \frac{w_1 + \mu_1}{a_1} - w_2 \right) \right) \cot \left( \pi \left( z - \frac{w_1 + \mu_1}{a_1} \right) \right) \\ &+ \frac{1}{a_2} \sum_{\mu_2=0}^{a_2-1} (-1)^{\mu_2} \cot \left( \pi \left( a_1 \frac{w_2 + \mu_2}{a_2} - w_1 \right) \right) \cot \left( \pi \left( z - \frac{w_2 + \mu_2}{a_2} \right) \right). \end{aligned} \tag{4.16}$$

(2)  $(K_1, K_2, J) = (I, II, II)$  case.

$$\begin{aligned}
& \cot \pi(a_1 z - w_1) \csc \pi(a_2 z - w_2) \\
&= -\frac{(-1)^{a_2(A_1 w_2 + A_2 w_1) - w_2}}{a_1 a_2} \delta_{\mathbb{Z}}(a_1 w_2 - a_2 w_1) \csc^{(1)}(\pi(z - (A_1 w_2 + A_2 w_1))) \\
&+ \frac{1}{a_1} \sum_{\mu_1=0}^{a_1-1}{}' \csc\left(\pi\left(a_2 \frac{w_1 + \mu_1}{a_1} - w_2\right)\right) \csc\left(\pi\left(z - \frac{w_1 + \mu_1}{a_1}\right)\right) \\
&+ \frac{1}{a_2} \sum_{\mu_2=0}^{a_2-1}{}' (-1)^{\mu_2} \cot\left(\pi\left(a_1 \frac{w_2 + \mu_2}{a_2} - w_1\right)\right) \csc\left(\pi\left(z - \frac{w_2 + \mu_2}{a_2}\right)\right). \tag{4.17}
\end{aligned}$$

(3)  $(K_1, K_2, J) = (II, II, I)$  case.

$$\begin{aligned}
& \csc \pi(a_1 z - w_1) \csc \pi(a_2 z - w_2) \\
&= -\frac{(-1)^{(a_1 + a_2)(A_1 w_2 + A_2 w_1) - (w_1 + w_2)}}{a_1 a_2} \delta_{\mathbb{Z}}(a_1 w_2 - a_2 w_1) \cot^{(1)}(\pi(z - (A_1 w_2 + A_2 w_1))) \\
&+ \frac{1}{a_1} \sum_{\mu_1=0}^{a_1-1}{}' (-1)^{\mu_1} \csc\left(\pi\left(a_2 \frac{w_1 + \mu_1}{a_1} - w_2\right)\right) \cot\left(\pi\left(z - \frac{w_1 + \mu_1}{a_1}\right)\right) \\
&+ \frac{1}{a_2} \sum_{\mu_2=0}^{a_2-1}{}' (-1)^{\mu_2} \csc\left(\pi\left(a_1 \frac{w_2 + \mu_2}{a_2} - w_1\right)\right) \cot\left(\pi\left(z - \frac{w_2 + \mu_2}{a_2}\right)\right). \tag{4.18}
\end{aligned}$$

(4)  $(K_1, K_2, J) = (II, II, II)$  case.

$$\begin{aligned}
& \csc \pi(a_1 z - w_1) \csc \pi(a_2 z - w_2) \\
&= -\frac{(-1)^{(a_1 + a_2)(A_1 w_2 + A_2 w_1) - (w_1 + w_2)}}{a_1 a_2} \delta_{\mathbb{Z}}(a_1 w_2 - a_2 w_1) \csc^{(1)}(\pi(z - (A_1 w_2 + A_2 w_1))) \\
&+ \frac{1}{a_1} \sum_{\mu_1=0}^{a_1-1}{}' (-1)^{\mu_1} \csc\left(\pi\left(a_2 \frac{w_1 + \mu_1}{a_1} - w_2\right)\right) \csc\left(\pi\left(z - \frac{w_1 + \mu_1}{a_1}\right)\right) \\
&+ \frac{1}{a_2} \sum_{\mu_2=0}^{a_2-1}{}' (-1)^{\mu_2} \csc\left(\pi\left(a_1 \frac{w_2 + \mu_2}{a_2} - w_1\right)\right) \csc\left(\pi\left(z - \frac{w_2 + \mu_2}{a_2}\right)\right). \tag{4.19}
\end{aligned}$$

(4.16), (4.17), (4.18) and (4.19) are generalizations of (1.2), (1.3), (1.4) and (1.5) respectively. Actually, by putting  $w_1 = w_2 = 0$ , our results become Fukuhara's formulas.

**Theorem 4.9.** (1)

$$\begin{aligned}
& a_2 \sum_{\mu_1=0}^{a_1-1}{}' (-1)^{\mu_1 \delta_{K_1, II}} \varphi_1^{(K_2)}\left(a_2 \frac{w_1 + \mu_1}{a_1} - w_2\right) \\
&+ a_1 \sum_{\mu_2=0}^{a_2-1}{}' (-1)^{\mu_2 \delta_{K_2, II}} \varphi_1^{(K_1)}\left(a_1 \frac{w_2 + \mu_2}{a_2} - w_1\right) = 0. \tag{4.20}
\end{aligned}$$

(2) For any  $\mu \in \mathbb{Z}_{\geq 0}$  and  $z_0 \in \mathfrak{R}$ ,

$$\begin{aligned}
& a_2 \sum_{\mu_1=0}^{a_1-1} (-1)^{\mu_1 \delta_{K_1, II}} \varphi_1^{(K_2)} \left( a_2 \frac{w_1 + \mu_1}{a_1} - w_2 \right) A_\mu^{(J)} \left( z_0; 1, 1, \frac{w_1 + \mu_1}{a_1} \right) \\
& + a_1 \sum_{\mu_2=0}^{a_2-1} (-1)^{\mu_2 \delta_{K_2, II}} \varphi_1^{(K_1)} \left( a_1 \frac{w_2 + \mu_2}{a_2} - w_1 \right) A_\mu^{(J)} \left( z_0; 1, 1, \frac{w_2 + \mu_2}{a_2} \right) \\
& = \pi^2 a_1 a_2 \delta_{K_1, I} \delta_{K_2, I} \delta_{\mu, 0} \\
& - \delta_{\mathbb{Z}}(a_1 w_2 - a_2 w_1) \operatorname{sgn}_2^{(K_1, K_2)}((a_1, a_2), (w_1, w_2), (A_1, A_2)) A_\mu^{(J)}(z_0; 1, 2, A_1 w_2 + A_2 w_1) \\
& + \operatorname{sgn}^{(K_1)}(z_0; a_1, w_1) A_{\mu+1}^{(K_2)}(z_0; a_2, 1, w_2) + \operatorname{sgn}^{(K_2)}(z_0; a_2, w_2) A_{\mu+1}^{(K_1)}(z_0; a_1, 1, w_1) \\
& + \sum_{\nu=0}^{\mu} A_\nu^{(K_1)}(z_0; a_1, 1, w_1) A_{\mu-\nu}^{(K_2)}(z_0; a_2, 1, w_2). \tag{4.21}
\end{aligned}$$

## 5 Concluding remarks

We give Theorems 3.1, 3.2, which include as special cases reciprocity laws of various generalized Dedekind sums. Finally, as a future work, we raise a problem for an elliptic analogue of our main results.

Fukuhara and Yui derived the following formula in [6]. Fix a complex number  $\tau$  with positive imaginary part. We put

$$\begin{aligned}
\wp(z, \tau) &:= \frac{1}{z^2} + \sum_{\substack{\gamma \in \mathbb{Z} + \mathbb{Z}\tau \\ \gamma \neq 0}} \left\{ \frac{1}{(z - \gamma)^2} - \frac{1}{\gamma^2} \right\}, \\
\wp(z, \tau) &:= \sqrt{\wp(z, \tau) - \wp\left(\frac{1}{2}, \tau\right)} = \frac{1}{z} - \sum_{\nu \geq 0} \alpha_{\nu+1}(\tau) z^\nu.
\end{aligned}$$

If  $p$  and  $q$  are relatively prime and  $p + q$  is odd, then

$$\begin{aligned}
\varphi(pz, \tau) \varphi(qz, \tau) &= -\frac{1}{pq} \varphi'(z, \tau) \\
&+ \frac{1}{p} \sum_{\substack{\mu, \lambda=0 \\ (\mu, \lambda) \neq (0, 0)}}^{p-1} \varphi\left(\frac{q(\mu + \lambda\tau)}{p}, \tau\right) \varphi\left(z - \frac{\mu + \lambda\tau}{p}, \tau\right) \\
&+ \frac{1}{q} \sum_{\substack{\mu, \lambda=0 \\ (\mu, \lambda) \neq (0, 0)}}^{q-1} \varphi\left(\frac{p(\mu + \lambda\tau)}{q}, \tau\right) \varphi\left(z - \frac{\mu + \lambda\tau}{q}, \tau\right). \tag{5.1}
\end{aligned}$$

This formula can be regarded as an elliptic analogue of (1.1).

Further, Egami [3] provided the following reciprocity law which is an elliptic analogue of Zagier's reciprocity laws (4.12). If  $a_1, \dots, a_r \in \mathbb{Z}_{\geq 0}$  are relatively prime and  $a_1 + \dots + a_r$  is even, then

$$\sum_{l=1}^r \sum_{\substack{\mu_l, \lambda_l=0 \\ (\mu_l, \lambda_l) \neq (0,0)}}^{a_l-1} (-1)^{\lambda_l} \prod_{1 \leq u \neq l \leq r} \left\{ \varphi \left( a_u \frac{\mu_l + \lambda_l \tau}{a_l}, \tau \right) a_u \right\} = -M(\tau; \mathbf{a}), \quad (5.2)$$

where

$$M(\tau; \mathbf{a}) := \sum_{N=1}^{r-1} \sum_{1 \leq \lambda_1 < \dots < \lambda_N \leq r} \sum_{\substack{r-1-N = \sum_{k=1}^N \nu_{\lambda_k}, \\ \nu_{\lambda_1}, \dots, \nu_{\lambda_N} \geq 0}} (-1)^N \prod_{u=1}^N \{ \alpha_{\nu_u+1}(\tau) a_u^{\nu_u+1} \}.$$

In this article, we obtain a generalization of (1.1) and (4.12). Therefore, we naturally consider the following problem.

**Problem 5.1.** *Give an elliptic analogue of Theorems 3.1 and 3.2.*

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